

MEMORYLESS NEAR-COLLISIONS, REVISITED

MARIO LAMBERGER AND ELMAR TEUFL

ABSTRACT. In this paper we discuss the problem of generically finding near-collisions for cryptographic hash functions in a memoryless way. A common approach is to truncate several output bits of the hash function and to look for collisions of this modified function. In two recent papers, an enhancement to this approach was introduced which is based on classical cycle-finding techniques and covering codes. This paper investigates two aspects of the problem of memoryless near-collisions. Firstly, we give a full treatment of the trade-off between the number of truncated bits and the success-probability of the truncation based approach. Secondly, we demonstrate the limits of cycle-finding methods for finding near-collisions by showing that, opposed to the collision case, a memoryless variant cannot match the query-complexity of the “memory-full” birthday-like near-collision finding method.

1. INTRODUCTION

The field of hash function research has developed significantly in the light of the attacks on some of the most frequently used hash functions like MD4, MD5 and SHA-1. As a consequence, academia and industry started to evaluate alternative hash functions, *e.g.* in the SHA-3 initiative organized by NIST [15]. During this ongoing evaluation, not only the three classical security requirements *collision resistance*, *preimage resistance* and *second preimage resistance* are considered. Researchers look at (semi-)free-start collisions, near-collisions, distinguishers, etc. A ‘behavior different from that expected of a random oracle’ for the hash function is undesirable as are weaknesses that are demonstrated only for the compression function and not for the full hash function.

Coding theory and hash function cryptanalysis have gone hand in hand for quite some time now, where a crucial part of the attacks is based on the search for low-weight code words in a linear code (*cf.* [2, 4, 17] among others). In this paper, we want to elaborate on a newly proposed application of coding theory to hash function cryptanalysis. In [12, 13], it is demonstrated how to use covering codes to find near-collisions for hash functions in a memoryless way. We also want to refer to the recent paper [8] which considers similar concepts from the viewpoint of locality sensitive hashing.

In all of the following, we will work with binary values, where we identify $\{0, 1\}^n$ with \mathbb{Z}_2^n . Let “+” denote the n -bit exclusive-or operation. The Hamming weight of a vector $v \in \mathbb{Z}_2^n$ is denoted by $w(v) = |\{i \mid v_i = 1\}|$ and the Hamming distance of two vectors by $d(u, v) = w(u + v)$. The Handbook of Applied Cryptography [14, page 331] defines *near-collision resistance* of a hash function H as follows:

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Definition 1 (Near-Collision Resistance). It should be hard to find any two inputs m, m^* with $m \neq m^*$ such that $H(m)$ and $H(m^*)$ differ in only a small number of bits:

$$d(H(m), H(m^*)) \leq \epsilon. \quad (1)$$

For ease of later use we also give the following definition:

Definition 2. A message pair m, m^* with $m \neq m^*$ is called an ϵ -near-collision for H if (1) holds.

Collisions can be considered a special case of near-collisions with the parameter $\epsilon = 0$. The generic method for finding collisions for a given hash function is based on the *birthday paradox* and attributed to Yuval [22]. There are well established cycle-finding techniques (due to Floyd, Brent, Nivasch, cf. [3, 11, 16]) that remove the memory requirements from an attack based on the birthday paradox (see also [20]). These methods work by repeated iteration of the underlying hash function where in all of these applications the function is considered to behave like a random mapping (cf. [7, 9]).

In [12, 13], the question is raised whether or not the above mentioned cycle-finding techniques are also applicable to the problem of finding near-collisions. We now briefly summarize the ideas of [12, 13].

Since Definitions 1 and 2 include collisions as well, the task of finding near-collisions is easier than finding collisions. We now want to have a look at generic methods to construct near-collisions which are more efficient than the generic methods to find collisions.

In the following, let $B_r(x) := \{y \in \mathbb{Z}_2^n \mid d(x, y) \leq r\}$ denote the *Hamming ball* (or *Hamming sphere*) around x of radius r . Furthermore, we denote by $S_n(r) := |B_r(x)| = \sum_{i=0}^r \binom{n}{i}$ the cardinality of any n -dimensional Hamming ball of radius r .

A simple adaption of the classical table-based birthday attack for finding ϵ -near-collisions is to start with an empty table, randomly select a message m and compute $H(m)$ and then test whether the table contains an entry $(H(m) + \delta, m^*)$ for some $\delta \in B_\epsilon(0)$ and arbitrary m^* . If so, the pair (m, m^*) is an ϵ -near-collision. If not, $(H(m), m)$ is added to the table and repeat. Then, we know the following:

Lemma 1 ([12]). *Let H be an n -bit hash function. If we assume that H acts like a random mapping, the average number of messages that we need to hash and store in a table-based birthday-like attack before we find an ϵ -near-collision is $O(2^{n/2} S_n(\epsilon)^{-1/2})$.*

Remark 1. We want to note that in this paper we are measuring the complexity of a problem by counting (hash) function invocations. This constitutes an adequate measure in the case of the memoryless algorithms in this paper, however the real computational complexity of the table-based algorithm above is dominated by the memory access, as the problem of searching for an ϵ -near-collision in the table is much harder than testing for a collision.

The first straight-forward approach to apply the cycle-finding algorithms to the problem of finding near-collisions is a truncation based approach.

Lemma 2. *Let H be an n -bit hash function. Let $\tau_\epsilon: \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^{n-\epsilon}$ be a map that truncates ϵ bits from its input at predefined positions. If we assume that $\tau_\epsilon \circ H$ acts like a random mapping, we can apply a cycle-finding algorithm to the map $\tau_\epsilon \circ H$ to find an ϵ -near-collision in a memoryless way with an expected complexity of about $2^{(n-\epsilon)/2}$.*

Proof. Under the assumptions of the lemma, the results from [7, 9] are applied to a random mapping with output length $n - \epsilon$. \square

2. A THOROUGH ANALYSIS OF THE TRUNCATION APPROACH

As indicated in [12], a simple idea to improve the truncation based approach is to truncate more than ϵ bits. That is, in order to find an ϵ -near-collision we simply truncate μ bits with $\mu > \epsilon$. A cycle-finding method applied to $\tau_\mu \circ H$ has an expected complexity of $2^{(n-\mu)/2}$ and deterministically finds two messages m, m^* such that $d(H(m), H(m^*)) \leq \mu$. However, we can look at the probability that these two messages m, m^* satisfy $d(H(m), H(m^*)) \leq \epsilon$ which is $2^{-\mu} \sum_{i=0}^{\epsilon} \binom{\mu}{i} = 2^{-\mu} S_\mu(\epsilon)$.

For a truly memoryless approach, multiple runs of the cycle-finding algorithm are interpreted as independent events. Therefore, the expected complexity to find an ϵ -near-collision can be obtained as the product of the expected complexity to find a cycle, and the expected number of repetitions of the cycle-finding algorithm, *i.e.* the reciprocal value of the probability that a single run finds an ϵ -near-collision. In other words, we end up with an expected complexity of

$$2^{(n+\mu)/2} S_\mu(\epsilon)^{-1} = 2^{(n+\mu)/2} \left(\sum_{i=0}^{\epsilon} \binom{\mu}{i} \right)^{-1} \quad (2)$$

Remark 2. In [12], the above approach was already proposed with $\mu = 2\epsilon + 1$. In this case (2) results in a complexity of

$$2^{(n+2\epsilon+1)/2} S_{2\epsilon+1}(\epsilon)^{-1} = 2^{(n+1)/2-\epsilon},$$

which clearly improves upon Lemma 2. Here we have used that $S_{2\epsilon+1}(\epsilon) = \frac{1}{2} S_{2\epsilon+1}(2\epsilon+1) = 2^{2\epsilon}$.

An interesting question that now arises is to find the number of truncated bits μ that constitutes the best trade-off between a larger μ , *i.e.* a faster cycle-finding part, and a higher number of repetitions for this probabilistic approach. In other words, we would like to determine the value of μ which minimizes (2) for a given ϵ . Analogously, we can search for an integer $\mu > \epsilon$ such that for a given ϵ the expression $2^{-\mu/2} S_\mu(\epsilon)$ is maximized. For small values of ϵ , values for μ were already computed in [12] by an exhaustive search. In this section, we want to solve this problem analytically.

We first show a result that tells us something about the behavior of the sequence of real numbers

$$a_\mu := 2^{-\mu/2} S_\mu(\epsilon) = 2^{-\mu/2} \sum_{i=0}^{\epsilon} \binom{\mu}{i}. \quad (3)$$

We want to note that based on the origin of the problem, we are only interested in values a_μ for $\mu > \epsilon$. Our analysis is still valid starting with $\mu = 1$. We will need the following two properties of sequences:

Definition 3. Let a_μ be a real-valued sequence.

- (i) A sequence a_μ is called *unimodal* in μ , if there exists an index t such that $a_1 \leq a_2 \leq \dots \leq a_t$ and $a_t \geq a_{t+1} \geq a_{t+2} \geq \dots$. The index t is called a *mode* of the sequence.
- (ii) A sequence a_μ is called *log-concave*, if $a_\mu^2 \geq a_{\mu-1} a_{\mu+1}$ holds for every μ . If \geq is replaced by $>$, we speak of a *strictly log-concave* sequence.

Lemma 3. *The sequence a_μ defined in (3) is strictly log-concave and therefore also unimodal.*

Proof. It is a well known fact that a log-concave sequence is also unimodal, cf. for example [18]. So in order to show that (3) is strictly log-concave we have to show that for any $\epsilon \geq 1$,

$$\sum_{i=0}^{\epsilon} \sum_{j=0}^{\epsilon} \binom{\mu}{i} \binom{\mu}{j} > \sum_{i=0}^{\epsilon} \sum_{j=0}^{\epsilon} \binom{\mu-1}{i} \binom{\mu+1}{j} \quad (4)$$

holds. By using the recursion for the binomial coefficient twice, we can transform the inequality (4) into

$$\sum_{i=0}^{\epsilon} \sum_{j=0}^{\epsilon} \left[\binom{\mu-1}{i} + \binom{\mu-1}{i-1} \right] \binom{\mu}{j} > \sum_{i=0}^{\epsilon} \sum_{j=0}^{\epsilon} \binom{\mu-1}{i} \left[\binom{\mu}{j} + \binom{\mu}{j-1} \right],$$

which boils down to the inequality

$$\binom{\mu}{\epsilon} \sum_{i=0}^{\epsilon-1} \binom{\mu-1}{i} > \binom{\mu-1}{\epsilon} \sum_{i=0}^{\epsilon-1} \binom{\mu}{i}.$$

By direct computation using the definition of the binomial coefficient, it is easy to see that each summand on the left is strictly larger than the respective summand on the right, simply because $\epsilon > i$. \square

The strict log-concavity guarantees us the existence of at most two adjacent indices for which the sequence a_μ attains its global maximum. But if there would be an index t , such that $a_t = a_{t+1}$ is maximal, the definition of the sequence a_μ in (3) shows that this would imply the existence of two positive integers a, b such that $a = \sqrt{2}b$, which is clearly not possible. Therefore, the mode of the sequence is indeed unique.

In order to find the mode of a_μ , we have to investigate some properties of truncated sums of binomial coefficients. There are well known bounds for the sum $S_\mu(\epsilon)$, which yield upper and lower bounds for the optimal value of μ . As we are interested in an asymptotically correct approximation for the optimal μ , we need to derive an asymptotic expansion of $S_\mu(\epsilon)$ which seems to be hard to find in the literature. Notationally, we use $f(\mu) \sim g(\mu)$ if $\lim_{\mu \rightarrow \infty} f(\mu)/g(\mu) = 1$ and $f(\mu) \asymp g(\mu)$ if there exist positive c_1, c_2, μ_0 such that $c_1 \cdot |g(\mu)| \leq |f(\mu)| \leq c_2 \cdot |g(\mu)|$ for all $\mu \geq \mu_0$.

Proposition 1. *Let $S_\mu(\epsilon) = \sum_{k=0}^{\epsilon} \binom{\mu}{k}$ and define $\alpha := \frac{\epsilon}{\mu}$. If we assume, that there exist constants c_1, c_2 such that $0 < c_1 \leq \alpha \leq c_2 < \frac{1}{2}$, then we have*

$$S_\mu(\epsilon) = \binom{\mu}{\epsilon} \cdot \left(\frac{\mu - \epsilon}{\mu - 2\epsilon} - \frac{2\epsilon(\mu - \epsilon)}{(\mu - 2\epsilon)^3} + O(\mu^{-2}) \right), \quad (5)$$

for $\epsilon, \mu \rightarrow \infty$ and thus

$$S_\mu(\epsilon) \sim \frac{\mu - \epsilon}{\mu - 2\epsilon} \cdot \binom{\mu}{\epsilon}.$$

Proof. For $k \leq \epsilon$ we have

$$\binom{\mu}{k} = \binom{\mu}{\epsilon} \prod_{i=0}^{\epsilon-k-1} \frac{\epsilon - i}{\mu - k - i} \leq \binom{\mu}{\epsilon} \cdot \left(\frac{\epsilon}{\mu - \epsilon} \right)^{\epsilon-k}. \quad (6)$$

Because of the requirements in the proposition we have

$$\frac{\epsilon}{\mu - \epsilon} = \frac{\alpha}{1 - \alpha} \leq \frac{c_2}{1 - c_2} < 1.$$

For sake of notation we set $\beta := \frac{\alpha}{1 - \alpha}$ and $c := \frac{c_2}{1 - c_2}$. This then leads to

$$\begin{aligned} \binom{\mu}{\epsilon} &\leq S_\mu(\epsilon) \leq \binom{\mu}{\epsilon} \sum_{k=0}^{\epsilon} \left(\frac{\epsilon}{\mu - \epsilon} \right)^{\epsilon - k} \\ &\leq \binom{\mu}{\epsilon} \sum_{j=0}^{\infty} \left(\frac{\epsilon}{\mu - \epsilon} \right)^j = \frac{\mu - \epsilon}{\mu - 2\epsilon} \cdot \binom{\mu}{\epsilon} \leq \frac{1}{1 - c} \cdot \binom{\mu}{\epsilon}. \end{aligned} \quad (7)$$

From equation (7) we learn that $S_\mu(\epsilon) \asymp \binom{\mu}{\epsilon}$.

The following can be seen as a discrete version of Laplace's method to approximate integrals (cf. [6]).

$$S_\mu(\epsilon) = \sum_{k=0}^{\epsilon} \binom{\mu}{k} = \sum_{0 \leq k \leq \epsilon - r} \binom{\mu}{k} + \sum_{\epsilon - r < k \leq \epsilon} \binom{\mu}{k} = S_\mu(\epsilon - r) + \sum_{0 \leq k < r} \binom{\mu}{\epsilon - k},$$

where $r = r(\mu)$ is such that $r = o(\mu)$ for $\mu \rightarrow \infty$. We will determine r later.

Because of (6) and (7) we obtain

$$S_\mu(\epsilon - r) \asymp \binom{\mu}{\epsilon - r} = \binom{\mu}{\epsilon} \cdot O(c^r).$$

This implies

$$S_\mu(\epsilon) = \binom{\mu}{\epsilon} \cdot \left(\sum_{0 \leq k < r} \prod_{i=0}^{k-1} \frac{\epsilon - i}{\mu - \epsilon + k - i} + O(c^r) \right).$$

We now have a closer look at the product above:

$$\prod_{i=0}^{k-1} \frac{\epsilon - i}{\mu - \epsilon + k - i} = \exp \left(\sum_{i=0}^{k-1} \log \frac{\alpha - \frac{i}{\mu}}{1 - \alpha + \frac{k}{\mu} - \frac{i}{\mu}} \right).$$

For x, y close to 0 we have

$$\log \frac{\alpha + x}{1 - \alpha + y} = \log \beta + \frac{1}{\alpha} \cdot x - \frac{1}{1 - \alpha} \cdot y + O(x^2 + y^2).$$

Since $0 \leq i < k < r$ and $r = o(\mu)$ we conclude

$$\log \frac{\alpha - \frac{i}{\mu}}{1 - \alpha + \frac{k}{\mu} - \frac{i}{\mu}} = \log \beta - \frac{1}{(1 - \alpha)} \cdot \frac{k}{\mu} - \frac{(1 - 2\alpha)}{\alpha(1 - \alpha)} \cdot \frac{i}{\mu} + O\left(\frac{k^2}{\mu^2}\right),$$

where the error term is uniform in $0 \leq k < r$. With this we get

$$\begin{aligned} \prod_{i=0}^{k-1} \frac{\epsilon - i}{\mu - \epsilon + k - i} &= \beta^k \exp \left(\frac{1 - 2\alpha}{2\alpha(1 - \alpha)} \cdot \frac{k}{\mu} - \frac{1}{2\alpha(1 - \alpha)} \cdot \frac{k^2}{\mu} + O\left(\frac{k^3}{\mu^2}\right) \right) \\ &= \beta^k \left(1 + \frac{1 - 2\alpha}{2\alpha(1 - \alpha)} \cdot \frac{k}{\mu} - \frac{1}{2\alpha(1 - \alpha)} \cdot \frac{k^2}{\mu} + O\left(\frac{k^3}{\mu^2}\right) \right). \end{aligned}$$

In total we obtain that $S_\mu(\epsilon)/\binom{\mu}{\epsilon}$ is equal to

$$\sum_{0 \leq k < r} \beta^k \left(1 + \frac{1-2\alpha}{2\alpha(1-\alpha)} \cdot \frac{k}{\mu} - \frac{1}{2\alpha(1-\alpha)} \cdot \frac{k^2}{\mu} + O\left(\frac{k^3}{\mu^2}\right) \right)$$

up to an error term which is bounded by $O(c^r)$. Since

$$\sum_{0 \leq k < r} \beta^k \cdot \frac{k^3}{\mu^2} = O(\mu^{-2})$$

and

$$\sum_{k \geq r} \beta^k \left(1 + \frac{1-2\alpha}{2\alpha(1-\alpha)} \cdot \frac{k}{\mu} - \frac{1}{2\alpha(1-\alpha)} \cdot \frac{k^2}{\mu} \right) = O(r^2 c^r),$$

it follows that $S_\mu(\epsilon)/\binom{\mu}{\epsilon}$ is equal to

$$\sum_{k \geq 0} \beta^k \left(1 + \frac{1-2\alpha}{2\alpha(1-\alpha)} \cdot \frac{k}{\mu} - \frac{1}{2\alpha(1-\alpha)} \cdot \frac{k^2}{\mu} \right) + O(\mu^{-2} + r^2 c^r).$$

Simplifying the infinite sum above yields

$$S_\mu(\epsilon) = \binom{\mu}{\epsilon} \cdot \left(\frac{1-\alpha}{1-2\alpha} - \frac{2\alpha(1-\alpha)}{(1-2\alpha)^3} \cdot \frac{1}{\mu} + O(\mu^{-2} + r^2 c^r) \right).$$

We now choose $r = r(\mu) = (\log \mu)^2$, since then $r^2 c^r = o(\mu^{-2})$, which readily implies the statement using the definition of α . \square

The results of the Lem. 3 and Prop. 1 can now be combined in the following way. We are interested in the behavior of a_μ , that is,

$$a_\mu = 2^{-\mu/2} S_\mu(\epsilon) = 2^{-\mu/2} \sum_{i=0}^{\epsilon} \binom{\mu}{i}.$$

We have already seen that there will be a unique mode t for the sequence. Until this index, we have $a_{\mu+1}/a_\mu \geq 1$ and for all following values of μ , we have $a_{\mu+1}/a_\mu \leq 1$. If we evaluate the fraction, we get $a_{\mu+1}/a_\mu = S_{\mu+1}(\epsilon)/(\sqrt{2} S_\mu(\epsilon))$. From the recurrence relation of the binomial coefficient we get the analogous recurrence relation for $S_\mu(\epsilon)$, namely $S_{\mu+1}(\epsilon) = S_\mu(\epsilon) + S_\mu(\epsilon-1) = 2S_\mu(\epsilon) - \binom{\mu}{\epsilon}$. If we use this in the above equation we end up with

$$\frac{a_{\mu+1}}{a_\mu} = \sqrt{2} \left(1 - \frac{\binom{\mu}{\epsilon}}{2S_\mu(\epsilon)} \right) \quad (8)$$

If we now use the asymptotic expansion in (5) we can compute an approximation for $\mu = \mu(\epsilon)$ such that an optimum for (2) is found.

Theorem 2. *Let H be a hash function producing an n -bit hash value and let $\epsilon \geq 1$ be given. Let $\tau_\mu: \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^{n-\mu}$ be a map that truncates μ fixed bits from an n -bit value, and suppose we apply a cycle-finding algorithm to $\tau_\mu \circ H$, which is assumed to act like a random mapping. Then, there exists a unique optimal choice $\mu = \mu(\epsilon) > \epsilon$ to find an ϵ -near-collision. For large ϵ , we have*

$$\mu(\epsilon) = (2 + \sqrt{2})(\epsilon - 1) + O(\epsilon^{-1}). \quad (9)$$

Proof. Substituting the lower bound

$$S_\mu(\epsilon) \geq \binom{\mu}{\epsilon} + \binom{\mu}{\epsilon-1} = \frac{\mu+1}{\mu+1-\epsilon} \binom{\mu}{\epsilon}$$

and the upper bound of (7) in (8) implies that the mode t of the sequence a_μ is bounded by $(1 + \sqrt{2})\epsilon - 1 \leq t \leq (2 + \sqrt{2})\epsilon$. For values of μ in the domain above we may use Prop. 1, since the quotient ϵ/μ is easily seen to be bounded in the right way. Furthermore, $\mu \asymp \epsilon$ and $\mu - 2\epsilon \asymp \epsilon$. For large values of ϵ we infer from

$$S_\mu(\epsilon) = \binom{\mu}{\epsilon} \cdot \left(\frac{\mu - \epsilon}{\mu - 2\epsilon} + O(\epsilon^{-1}) \right),$$

that the mode t must satisfy the equation

$$1 = (2 - \sqrt{2}) \left(\frac{t - \epsilon}{t - 2\epsilon} + O(\epsilon^{-1}) \right).$$

Solving this equation yields $t = (2 + \sqrt{2})\epsilon + O(1)$. Now let us try to obtain further terms of the asymptotic expansion of t using bootstrapping (see for instance [6]). Using the full strength of Prop. 1 implies that the equation

$$1 = (2 - \sqrt{2}) \left(\frac{t - \epsilon}{t - 2\epsilon} - \frac{2\epsilon(t - \epsilon)}{(t - 2\epsilon)^3} + O(\epsilon^{-2}) \right)$$

must be satisfied by the mode t . Using the ansatz $t = (2 + \sqrt{2})\epsilon + r$, where $r = O(1)$, yields

$$2(1 + \sqrt{2}) \left((3 - 2\sqrt{2})r + (2 - \sqrt{2}) \right) \epsilon^2 + O(\epsilon) = 0.$$

Hence we get $r = -(2 + \sqrt{2}) + O(\epsilon^{-1})$ and $t = (2 + \sqrt{2})(\epsilon - 1) + O(\epsilon^{-1})$ which corresponds to $\mu(\epsilon)$. \square

We want to note that in both, Prop. 1 and Th. 2, it is possible to compute an arbitrary number of terms of the asymptotic expansions (5) and (9). We end this section with Table 1 demonstrating the quality of the approximation of (9). The actual values for $\mu(\epsilon)$ are produced by an exhaustive search and for simplicity, (9) is replaced with $\lceil (2 + \sqrt{2})(\epsilon - 1) \rceil$.

TABLE 1. Comparison of $\mu(\epsilon)$ and $\mu^*(\epsilon) := \lceil (2 + \sqrt{2})(\epsilon - 1) \rceil$ for certain values of ϵ .

ϵ	1	2	3	4	...	8	9	10	...	98	99	100
$\mu(\epsilon)$	2	5	8	11	...	25	28	32	...	332	335	339
$\mu^*(\epsilon)$	0	4	7	11	...	24	28	31	...	332	335	339

3. LIMITATIONS OF MEMORYLESS NEAR-COLLISIONS

A drawback to the truncation based solution is of course that we can only find ϵ -near-collisions of a limited shape (depending on the fixed bit positions), so only a fraction of all possible ϵ -near-collisions can be detected, namely $S_\mu(\epsilon)/S_n(\epsilon)$.

To improve upon this, [12, 13] had the idea is to replace the projection τ_ϵ by a more complicated function g , where g is the decoding operation of a certain

covering code \mathcal{C} . Let $R = R(\mathcal{C})$ be the covering radius of a code \mathcal{C} , that is $R(\mathcal{C}) = \max_{x \in \mathbb{Z}_2^n} \min_{c \in \mathcal{C}} d(x, c)$.

Theorem 3 ([12]). *Let H be a hash function of output size n . Let \mathcal{C} be a covering code of the same length n , size K and covering radius $R(\mathcal{C})$ and assume there exists an efficiently computable map g such that $g: \mathbb{Z}_2^n \rightarrow \mathcal{C}$, where $x \mapsto c$ with $d(x, c) \leq R(\mathcal{C})$. If we further assume that $g \circ H$ acts like a random mapping, in the sense that the expected cycle and tail lengths are the same as for the iteration of a truly random mapping on a space of size K , then we can find $2R(\mathcal{C})$ -near-collisions for H with a complexity of about \sqrt{K} and with virtually no memory requirements.*

An extensive amount of work in the theory of covering codes is devoted to derive upper and lower bounds for K (when n and R are given) and to construct codes achieving these bounds (cf. [5, 19, 21]). The authors of [13] have investigated a class of efficient codes suitable for the approach outlined in Th. 3. The approach via covering codes constitutes an improvement over the purely truncation based approach. However, (depending on ϵ) the query-complexity of the approach outlined in Th. 3 is larger than the expected query-complexity of the table-based birthday method, cf. Lem. 1.

Remark 3. We briefly want to mention the possibility of considering a probabilistic version of the covering code approach in an analogous manner to the approach in Sec. 2. In other words, what is the probability to find a $(2R - 1)$ -near-collision if the covering radius is R ? This problem has also been studied in [12] with the outcome that in general, finding a closed expression like (2) is beyond reach. Numerical experiments for relevant values of n and $\epsilon = 2R$ show, that increasing the covering radius is rarely bringing an improvement. We use [12, Eq. (20)] together with the optimal solution from [13] to compute complexities for small values of ϵ in Table 3.

The limitations of the covering code approach are inherent to the *sphere covering bound*, which states that $K \geq 2^n / S_n(R)$ (cf. [5]). Since we use codes with covering radius R to find $2R$ -near-collisions, that is, $\epsilon = 2R$, the sphere covering bound implies that the size K of the code has to be larger than $K \geq 2^n / S_n(R) \gg 2^n / S_n(2R)$, where the latter would be the desired quantity to match the complexity of Lem. 1 to find an ϵ -near-collision.

In the following, we want to investigate, if there are other possibilities to choose a mapping g such that collisions for $g \circ H$ imply ϵ -near-collisions for H . In [12] it was shown, that the “perfect” mapping g is beyond reach:

Lemma 4 ([12]). *Let $\epsilon \geq 1$, let H be a hash function and let g be a function such that*

$$d(H(m), H(m^*)) \leq \epsilon \Leftrightarrow g(H(m)) = g(H(m^*))$$

holds. Then, g is a constant map and $d(H(m), H(m^)) \leq \epsilon$ for all m, m^* .*

So the best we can hope for is a mapping $g: \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^k$ that satisfies

$$g(y) = g(y') \Rightarrow d(y, y') \leq \epsilon, \quad (10)$$

for all $y, y' \in \mathbb{Z}_2^n$. If we recall the requirements of Th. 3, it was stated that $g \circ H$ should act like a random mapping in order to have the expected cycle and tail lengths of the iteration of $g \circ H$ to be the same as for a truly random mapping on a space of size 2^k .

We formalize this in the following lemma. For this, we assume that the hash function H acts like a random mapping from a large domain $D \simeq \mathbb{Z}_2^\ell$ to \mathbb{Z}_2^n (since most hash standards define a maximum input length). First, we need yet another definition:

Definition 4. Let D, I be finite domains. We call a function $g: D \rightarrow I$ *balanced*, if $|I|$ divides $|D|$ and for all $z \in I$ we have $|g^{-1}(z)| = |D|/|I|$.

Lemma 5. Let $H: \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^n$ be a random mapping. Furthermore, consider a function $g: \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^k$ with $k \leq n$. Then, g is balanced if and only if $g \circ H: \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^k$ is a random mapping.

Proof. Let g be balanced, that is, for all $z \in \mathbb{Z}_2^k$ we have $|g^{-1}(z)| = 2^{n-k}$. The sets $P_z := g^{-1}(z)$ for all $z \in \mathbb{Z}_2^k$ define a disjoint partition of \mathbb{Z}_2^n of size $|P_z| = 2^{n-k}$ and g is constant on each set P_z .

Now let H be drawn uniformly at random from the set of all functions $\mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^n$, that is, for any function $h: \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^n$ we have $\mathbb{P}(H = h) = 2^{-n2^\ell}$. For a given $h': \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^k$, we now want to compute the probability $\mathbb{P}(g \circ H = h')$, for which we get

$$\begin{aligned} \mathbb{P}(g \circ H = h') &= 2^{-n2^\ell} |\{h: \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^n \mid g(h(x)) = h'(x) \text{ for all } x \in \mathbb{Z}_2^\ell\}| \\ &= 2^{-n2^\ell} |\{h: \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^n \mid h(x) \in P_{h'(x)} \text{ for all } x \in \mathbb{Z}_2^\ell\}| \\ &= 2^{-n2^\ell} 2^{(n-k)2^\ell} = 2^{-k2^\ell}, \end{aligned} \quad (11)$$

because $|P_z| = 2^{n-k}$ for all z . In other words, $g \circ H$ is a random mapping.

Now assume that $g \circ H$ is a random mapping. That means, that for every $h': \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^k$ we have $\mathbb{P}(g \circ H = h') = 2^{-k2^\ell}$. This stays true, if we choose h' to be one of the 2^k constant functions. If we argue along the same lines as in (11), we get

$$2^{-k2^\ell} = 2^{-n2^\ell} |\{h: \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^n \mid g(h(x)) = c \text{ for all } x \in \mathbb{Z}_2^\ell\}|$$

for all $c \in \mathbb{Z}_2^k$. Again, with $P_c = g^{-1}(c)$, we have

$$2^{(n-k)2^\ell} = |\{h: \mathbb{Z}_2^\ell \rightarrow \mathbb{Z}_2^n \mid h(x) \in P_c \text{ for all } x \in \mathbb{Z}_2^\ell\}|.$$

This leaves us with $|P_c| = |g^{-1}(c)| = 2^{n-k}$ for all $c \in \mathbb{Z}_2^k$, and thus, g is balanced. \square

Lem. 5 teaches us, that in a memoryless near-collision algorithm based on the iteration of the concatenation of the hash function H and a function g , additionally to the requirement (10) we need g also to be balanced. In the remaining part of this section, we want to show that this limits our choices basically to the known candidates for g .

For the proof of the next proposition, we will need a lemma which goes back to a conjecture by Erdős. The solution of this problem by Kleitman in [10], was further investigated in [1]. Let $\text{diam}(A)$ be the diameter of a set $A \subset \mathbb{Z}_2^n$, i.e., $\text{diam}(A) := \max_{x,y \in A} d(x,y)$. We now collect the results of Th. 1 and Th. 2 of [1] in the following lemma:

Lemma 6. Let s be a non-negative integer.

- (i) The Hamming balls $B_s(x)$ for any $x \in \mathbb{Z}_2^n$ are the sets of maximal size among all sets $A \subset \mathbb{Z}_2^n$ with $\text{diam}(A) = 2s < n - 1$.
- (ii) The sets $B_s(x) \cup B_s(y)$ for any $x, y \in \mathbb{Z}_2^n$ with $d(x, y) = 1$ are sets of maximal size among all sets $A \subset \mathbb{Z}_2^n$ with $\text{diam}(A) = 2s + 1 < n - 1$.

With this auxiliary result, we can now formulate the main result of this section.

Theorem 4. *Let $1 \leq \epsilon < \frac{n}{2}$ be given and let $g: \mathbb{Z}_2^n \rightarrow I$ be a balanced function satisfying (10), that is,*

$$g(y) = g(y') \Rightarrow d(y, y') \leq \epsilon$$

for all $y, y' \in \mathbb{Z}_2^n$. Then, $|I|$ must satisfy

$$|I| \geq 2^n / S_n(\lceil \epsilon/2 \rceil). \quad (12)$$

Proof. In the proof of Lem. 5 we have seen, that the balancedness of g implies a disjoint partition $\bigcup_z P_z$ of \mathbb{Z}_2^n where the size of each set P_z is $2^n/|I|$. The sets P_z are exactly such that $g(x) = z$ for all $x \in P_z$. Taking property (10) into account, we need that $\text{diam}(P_z) \leq \epsilon$. Therefore, Lem. 6 teaches us that if ϵ is even, we have $2^n/|I| \leq S_n(\epsilon/2)$ and $2^n/|I| \leq S_n(\frac{\epsilon-1}{2}) + \binom{n-1}{(\epsilon-1)/2}$, for odd ϵ . Since $S_n(\frac{\epsilon-1}{2}) + \binom{n-1}{(\epsilon-1)/2} \leq S_n(\frac{\epsilon+1}{2})$, we can unify the expressions to (12). \square

As a consequence we get the following corollary:

Corollary 1. *Let H be an n -bit hash function, let $1 \leq \epsilon < \frac{n}{2}$ and let g be a balanced function satisfying (10). Then, the complexity to find an ϵ -near-collision by applying a cycle-finding algorithm to the concatenation $g \circ H$ is bounded from below by $\Omega(2^{n/2} S_n(\lceil \epsilon/2 \rceil)^{-1/2})$.*

4. CONCLUSION

At the moment, a lot of effort is dedicated to the cryptanalysis of concrete hash function designs. From a theoretical perspective it is still very important to investigate generic aspects of non-random properties of hash functions. In this paper, we have analyzed several aspects of the question of finding near-collisions in a memoryless way. This problem has recently been investigated in [12, 13]. All these methods rely on the application of a cycle-finding technique to an alteration (that is, concatenation with a new mapping) of the hash function. We have investigated in full detail the complexity of a probabilistic version of the simple truncation based approach. Furthermore, we have shown that the approach in general is limited in its capabilities, in the sense, that if g is such that finding a collision for $g \circ H$ implies a near-collision for H , the query-complexity of this approach is always higher than the query-complexity of a birthday-like method using a table of exponential size. A comparison of the known methods is compiled in Tables 2 and 3. It has to be noted that in practice the real complexity of the table-based method will be dominated by the table queries and not by the hash computations.

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TABLE 2. Methods for finding ϵ -near-collisions of an n -bit hash function H .

short explanation	memory	complexity	remarks
cycle finding approach applied to an ϵ -truncation of H	negligible (memory is only required for cycle finding)	$2^{(n-\epsilon)/2}$	<i>cf.</i> Lemma 2 and [9];
cycle finding approach applied to an $2\epsilon + 1$ -truncation of H	negligible (memory is only required for cycle finding)	$2^{(n+1)/2-\epsilon}$	<i>cf.</i> Remark 2 and [12]; (A) in Table 3;
cycle finding approach applied to an optimized μ -truncation of H ($\mu > \epsilon$)	negligible (memory is only required for cycle finding)	$2^{(n+\mu)/2} S_\mu(\epsilon)^{-1}$	optimal $\mu = \mu(\epsilon)$ is unique and $\mu \sim (2 + \sqrt{2})(\epsilon - 1)$, <i>cf.</i> Theorem 2; (B) in Table 3;
table based approach	a table of exponential size in n for the pairs $(m, H(m))$	$2^{n/2} S_n(\epsilon)^{-1/2}$	<i>cf.</i> Lemma 1 and [12]; (C) in Table 3;
coding based approach	negligible (memory is only required for coding and cycle finding)	for even $\epsilon = 2R$: $2^{(n-\ell R-r)/2}$, where $\ell := \lfloor \log_2(n/R + 1) \rfloor$, $r := \lfloor (n - R(2^\ell - 1))/2^\ell \rfloor$	<i>cf.</i> [12, 13]; (D) in Table 3; for odd ϵ the coding based approach for $\epsilon + 1$ is repeated until an ϵ -near-collision is found, <i>cf.</i> Remark 3;

TABLE 3. For given $\epsilon \in \{1, \dots, 8\}$ and hash length $n \in \{160, 256, 512\}$, the table compares the base-2 logarithms of the complexities (A) – (D) of Table 2, together with (E) which is the bound of Corollary 1.

ϵ	$n = 160$					$n = 256$					$n = 512$				
	(A)	(B)	(C)	(D)	(E)	(A)	(B)	(C)	(D)	(E)	(A)	(B)	(C)	(D)	(E)
1	79.5	79.4	76.3	81.9	76.3	127.5	127.4	124.0	130.4	124.0	255.5	255.4	251.5	258.9	251.5
2	78.5	78.5	73.2	76.5	76.3	126.5	126.5	120.5	124.0	124.0	254.5	254.5	247.5	251.5	251.5
3	77.5	77.5	70.3	77.5	73.2	125.5	125.5	117.3	125.4	120.5	253.5	253.5	243.8	253.4	247.5
4	76.5	76.4	67.7	74.0	73.2	124.5	124.4	114.3	121.0	120.5	252.5	252.4	240.3	248.0	247.5
5	75.5	75.2	65.2	74.0	70.3	123.5	123.2	111.5	121.7	117.3	251.5	251.2	237.0	249.1	243.8
6	74.5	74.1	62.8	71.5	70.3	122.5	122.1	108.8	118.5	117.3	250.5	250.1	233.8	245.0	243.8
7	73.5	72.9	60.6	71.3	67.7	121.5	120.9	106.2	118.5	114.3	249.5	248.9	230.7	245.5	240.3
8	72.5	71.7	58.5	69.5	67.7	120.5	119.7	103.7	116.0	114.3	248.5	247.7	227.7	242.0	240.3

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INSTITUTE FOR APPLIED INFORMATION PROCESSING AND COMMUNICATIONS, GRAZ UNIVERSITY OF TECHNOLOGY, INFFELDGASSE 16A, A-8010 GRAZ, AUSTRIA.

E-mail address: `mario.lamberger@iaik.tugraz.at`

MATHEMATISCHES INSTITUT, EBERHARD KARLS UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, D-72076 TÜBINGEN, GERMANY.

E-mail address: `elmar.teufl@uni-tuebingen.de`